2.41. Proofs, Theorems, and Inconsistency

We've stressed the marvelous correlation of the deductive system with the earlier semantic tests of validity – both approaches picking out exactly the same arguments as valid. And already that scales up to a method for showing that two sentences are **logically equivalent**. For logical equivalence is simply a matter of each sentence following validly from the other – which, in the present context, amounts to each sentence being deducible from the other.

But the semantic approach also picked out two special families of sentences: the **tautologies** and the **contradictions**; and the **inconsistency** found in contradictions was then extended to **sets of sentences**. Here we explore how these families of sentences are likewise picked out in the deductive system.

1. Proofs and Theorems. We recognized the following as an example of a **tautology**, or **logical truth**.[1]

> **P**: Lucretia passed Chemistry
>
> "Either Lucretia passed Chemistry or she didn't."
>
> $\mathbf{(P \vee {\sim}P)}$

Semantically, logical truths are true in every possible situation (every valuation). This sentence is true if Lucretia passed Chemistry (if "P" is true), but also if she didn't pass Chemistry (if "P" is false). Being **true regardless of the facts** about Lucretia and Chemistry is what makes this sentence a **logical** truth – a sentence true through its logical form alone. As the English example above illustrates, logical truths are true independent of the facts because they are so **uninformative** about those facts: if Lucretia is anxious to know whether she passed Chemistry, the above sentence tells her nothing.

Such truth-regardless-of-the-facts accounts for a peculiar feature noted about logical truths and arguments[2]: any argument with a logical truth as its conclusion is bound to be valid. Since such a conclusion is always true, it provides no

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[1] In 2.17.

[2] In 2.19.

opportunity for a validity counterexample. A logical truth is true in and of itself, no matter which premises the argument has.

That is a clue to the treatment of logical truths in deductive terms. Taking a logical truth as the conclusion of an argument which is valid regardless of what its premises are, we say that the deduction of such an ‘argument’ can be completed without appeal to the premises. That is: a logical truth is the ‘conclusion’ of an argument deducible **without premises**. Logical truths live up to their name by being deducible from the bare deductive apparatus alone – the inference rules and ID – without appeal to premises.

We call a deduction with no premises a **proof**. And any sentence for which there is such a proof is called a **theorem** of the deductive system.

We can construct a proof of the logical truth “$(P \lor \sim P)$,” showing that it’s a theorem of our deductive system. (We do the proofs here twice over – once without appeal to DeMorgan’s Law, once using it – to illustrate the economy DM brings.)

Without Demorgan’s Law

	Get: $(P \lor \sim P)$ (ID)	
1.	$\sim(P \lor \sim P)$	AID
	Get: P (ID)	
2.	$\sim P$	AID
3.	$(P \lor \sim P)$	2, $\lor$+
4.	$\sim(P \lor \sim P)$	1, R
5.	P	3, 5, 6, ID
6.	$(P \lor \sim P)$	5, $\lor$+
7.	$\sim(P \land Q)$	1, 6, ID

With Demorgan’s Law

	Get: $(P \lor \sim P)$ (ID)	
1.	$\sim(P \lor \sim P)$	AID
2.	$(P \land \sim P)$	1, DM
3.	$\sim P$	2, $\land$–
4.	$\sim P$	2, $\land$–
5.	$(P \lor \sim P)$	1, 3, 6, ID

It is a further happy feature of our deductive system that its theorems – the sentences it can prove, without appeal to premises – are exactly the logical truths

picked out by the semantics.[3] That means that we can show that a sentence is a (semantic) logical truth by constructing a proof of it; or that a sentence is a (provable) theorem by showing in the semantics that it's logically true.
Note that our deductive system couldn't have managed proofs until we added indirect deduction, since every inference rule requires at least one sentence as input; so without the AID that indirect deduction brings the proof would never get started. Allowing proofs is thus a further benefit of adding ID to the deductive system.

2. Inconsistent Sentences and Sets of Sentences. The deductive system can show that a single sentence is **inconsistent** – a **contradiction** – in various ways. **First approach**: since the negation of a contradiction is a logical truth, to show that a sentence is a contradiction it suffices to prove its negation.[4]

Second approach: deduce from the sentence in question a known contradiction – say, a sentence of the form "(● ∧ ~●)". For as single sentences go, only a contradiction will validly entail a contradiction.[5] So if we can deduce, e.g., "(P ∧ ~P)" from a sentence, we know that sentence is a contradiction.

(Note that "(P ∧ ~P)" is just the one-sentence counterpart to the sort of contradictory sentences which close an ID box. But with ID that is quarantined within the ID, never making it outside the ID box. In this contradiction test, by contrast, the contradictory conclusion does appear outside any ID box. Indeed, this test need not even appeal to ID.)

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[3] That **all** the logical truths of semantics are theorems of the deductive system is referred to as the **completeness** of the deductive system. That **only** the logical truths are theorems is referred to as the **soundness** of the deductive system. While the soundness and completeness of the deductive system can be proven in a variety of ways, such **metalogical** proofs lie outside the scope of this book.

[4] Thanks to the soundness and completeness of the deductive system, we know that (i) every contradiction has a provable negation, and (ii) only a contradiction has a provable negation. For (i): if some contradiction had an unprovable negation, that negation would be a logical truth which was unprovable. But the completeness of the deductive system rules that out. For (ii): if some non-contradictory sentence had a provable negation, then that sentence, as non-contradictory, would be true in some valuation, meaning its negation was false in that valuation, hence not a logical truth. In that case we would have a proof of a sentence which is not logically true. But the soundness of the deductive system rules that out.

[5] As noted in 2.18.1 Problem E.

For example, we can show deductively that “$\sim(P \lor \sim P)$” is a contradiction using either approach.

First Approach

Get: $\sim\sim(P \lor \sim P)$ (ID)

1.	$\sim\sim\sim(P \lor \sim P)$	AID
2.	$\sim(P \lor \sim P)$	1, $\sim-$
3.	$(P \land \sim P)$	2, DM
4.	$\sim P$	3, $\land-$
5.	$\sim P$	3, $\land-$
6.	$\sim\sim(P \lor \sim P)$	1, 4, 5, ID

Second Approach

1.	$\sim(P \lor \sim P)$	
	Get: $(P \land \sim P)$	
2.	$(\sim P \land \sim\sim P)$	1, DM
3.	$\sim P$	2, $\land-$
4.	$\sim\sim P$	2, $\land-$
5.	P	4, $\sim-$
6.	$(P \land \sim P)$	3, 5, $\land+$

There would thus seem little to recommend one approach over the other, as a demonstration of contradiction. But the second approach has the advantage of conveniently scaling up to test **sets of sentences** for inconsistency. On the first method we could translate any finite set of sentences into one long conjunction (true just when all the sentences in the set are true), and then show that the negation of that conjunction is a theorem. But if the set of sentences contains many sentences that negated conjunction will be a cumbersome chore to deal with.

By contrast, following the second method we list each sentence in the set on a separate line, as in the second approach, the deduction can proceed as usual, applying rules to individual premises immediately.

Summary: Theorems and Inconsistency

- Two formal sentences are **logically equivalent** if each is deducible from the other.
- A deduction of a sentence without use of premises is a **proof** of that sentence.
- A sentence which is provable is a **theorem**. Every theorem of this deductive system is a (semantic) **logical truth**.
- A sentence is a **contradiction** (is **inconsistent**) if (and only if) there is a proof of the negation of that sentence.
- A set of (one or more) sentences is **inconsistent** if a contradiction – for example, a sentence of the form "(● ∧ ~●)" – is deducible from those sentences.